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STABILITY ANALYSIS OF COMBINED CONTINUOUS AND DISCRETE SYSTEMS

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16. Abstract

A discrete, time-invariant system is represented by an equivalent continuous, periodic system. A combined continuous plant and discrete controller may then be formulated as a single continuous, periodic coefficient system. The exact stability solution for the combined system is obtained, as a matrix eigenvalue problem. Periodic system theory also gives some information about the type of instabilities which may be encountered in the combined plant and controller.

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Abstract

A discrete, time-invariant system is represented by an equivalent continuous, periodic system. A combined continuous plant and discrete controller may then be formulated as a single continuous, periodic coefficient system. The exact stability solution for the combined system is obtained, as a matrix eigenvalue problem. Periodic system theory also gives some information about the type of instabilities which may be encountered in the combined plant and controller.

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Introduction

When the control of a continuous plant is implemented using a discrete filter and controller, it is necessary to analyze the combined system, which is part continuous and part discrete. One approach is to approximate the system by one which is all continuous or all discrete, but such an approximation may either introduce instabilities which are not present in the real system, or miss the actual instabilities. This approach is probably satisfactory for analysis of the system performance, such as rms response, but a better method is desired for calculating the system stability.

An exact method is required for analyzing the stability of a combined discrete and continuous system. Two facts suggest that the method be sought using periodic system theory. The first is the similarity in the state transition matrices of discrete and periodic systems (as discussed below); the second is the similarity in the type of instabilities. A major concern with combined continuous and discrete systems is the possibility of an instability characterized by the frequency locked at $2\pi/T$ (where T is the sampling period), which is not seen if the states are only observed at the sampling times. Such an instability is characteristic of a periodic coefficient system. Clearly the overall model for the system must be continuous, so we seek an equivalent model for the discrete part of the system. Consider a linear, time-invariant discrete system, of the form:

$$x(k+1) = Fx(k) + Gv(k)$$

where F and G are constant matrices. The solution of this equation is

$$x(k) = F^{k-k_o}\, x(k_o) + \sum_{j=k_o}^{k-1} F^{k-1-j}\, Gv(j)$$

Periodic, Continuous Systems

To obtain the equivalent model for this discrete system, the results of periodic system theory are required. Consider a linear, periodic, continuous system:

$$\dot{x} = A(t)x + B(t)v$$

where A is a periodic matrix, $A(t+T) = A(t)$. For a general time-varying system, the solution is described by the state transition matrix Φ:

$$x(t) = \Phi(t,t_o)x(t_o) + \int_{t_o}^{t} \Phi(t,\tau)B(\tau)v(\tau)d\tau$$

where Φ is the solution of $\dot{\Phi} = A\Phi$, $\Phi(t_o,t_o) = I$. It may be shown [1] that for periodic coefficient equations, Φ takes the form:

$$\Phi(t,\ ) = P(t)e^{\beta(t-t_o)}P^{-1}(t_o)$$

The matrices β and P are defined by

$$\alpha = e^{\beta T} = \Phi(T,0)$$

$$P(t) = \Phi(t,0)\alpha^{-t/T} = \Phi(t,0)e^{-\beta t}$$

where α and β (the eigenvalues) are constant matrices, and P (the eigenvector matrix) is periodic. Writing $\alpha = S\Theta S^{-1}$, $\beta = S\Lambda S^{-1}$ (Θ and Λ being the eigenvalues of α and β respectively, so $\Lambda = \frac{1}{T}\ln\Theta$), the transient solution is

$$x = (PS)e^{\Lambda t}q(0) = \sum_i u_i(t)e^{\lambda_i t}q_i(0)$$

This is a modal expansion of the solution for x, with periodic eigenvectors u_i. Furthermore, $\dot{\Phi} = A\Phi$ gives a differential equation for P:

$$\dot{P} = AP - P\beta \quad , \quad P(0) = I,\ P(t+T) = P(t)$$

or

$$\dot{u}_i' = (A - \lambda_i I)u_i$$

Now a general result for periodic systems is that the state transition matrix over any period is related to the initial period by:

$$\Phi(t+NT, t_o) = \Phi(t, t_o)\,\alpha^N$$

It follows that the solution is defined entirely by α, plus the values of Φ over the single period $t = t_o$ to t_o+T. The solution of a periodic coefficient equation, evaluated at $t = kT$, is then:

$$x(kT) = \alpha^{k-k_o}\, x(k_oT) + \int_{k_oT}^{kT} \Phi(kT,\tau)B(\tau)v(\tau)d\tau$$

Finally, note that for the time-invariant case (A constant), the solution reduces to $\alpha = e^{AT}$ and $\Phi = e^{A(t-t_o)}$.

Equivalent Periodic and Discrete Systems

Comparing the solutions above for $x(k)$ of a discrete system and $x(kT)$ for a periodic, continuous system, the similarity mentioned in the introduction is apparent. Clearly for the equivalent periodic, continuous system we want $\alpha = F$. To complete the description of the solution, the state transition matrix is required over a single period. The obvious choice for a discrete system model is Φ = constant, hence

$$\Phi(t,0) = F^k, \qquad (k-1)T < t \leq kT$$

What does this state transition matrix imply for the system (i.e. A)? We have

$$P = \Phi(t,0)\,\alpha^{-t/T} = F^{-(t/T-k)}$$

so

$$\dot{P} = \left(\sum_{j=-\infty}^{\infty} \delta(t-jT)\right)(F-I) - P\beta$$

Then the differential equation for P gives

$$A = (\dot{P} + P\beta)P^{-1} = \left(\sum_{j=-\infty}^{\infty} \delta(t-jT)\right)(I - F^{-1})$$

Similarly the control matrix is chosen to be

$$B = \left(\sum_{j=-\infty}^{\infty} \delta(t-jT)\right) F^{-1}G$$

The solution of this periodic system is

$$x(t) = F^{m-k_o}\, x(k_oT) + \sum_{j=k_o}^{m-1} F^{m-1-j} Gv(jT), \qquad mT < t \leq (m+1)T$$

The solution at $t = kT$ is indeed then

$$x(kT) = F^{k-k_o}\, x(k_oT) + \sum_{j=k_o}^{k-1} F^{k-1-j}\, Gv(jT)$$

the same solution as the discrete system at the sample times (and constant in between).

Hence the equivalent continuous system for

$$x(k+1) = Fx(k) + Gv(k)$$

is

$$\dot{x} = \Big(\sum_{j=-\infty}^{\infty} \delta(t-jT) \Big) \big((I - F^{-1})x + F^{-1}Gv \big)$$

The periodic part of the equation coefficients is just a sequence of impulses, as might be expected of a discrete system.

Combined Continuous and Discrete System

The above result may now be applied to the stability analysis of a combined discrete and continous system. Consider the following plant and controller:

continuous plant	$\dot{x} = Fx + Gv$
measurement	$z = Hx$
discrete filter	$y(k+1) = Ay(k) + Bz(k)$
controller	$v(k) = Cy(k)$

The A/D conversion is part of the measurement stage; infinite word size is assumed. The D/A conversion here is a first order hold, so the control signal is constant between sampling times; the D/A filter can be considered part of the continuous system.

Using the results of the previous section, the discrete system is modeled by

$$\dot{y} = \left(\sum_{j=-\infty}^{\infty} \delta(t-jT) \right) \left((I - A^{-1})y + A^{-1}Bz \right)$$
$$v = Cy$$

Hence the complete coupled system is

$$\begin{pmatrix} x \\ y \end{pmatrix}^{\cdot} = \begin{bmatrix} F & GC \\ (\Sigma\delta)A^{-1}BH & (\Sigma\delta)(I - A^{-1}) \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

The time-invariant continuous plant plus discrete controller is described by this continous, periodic system. Two things are gained by this formulation: first, a procedure for calculating the exact stability; and second, information about the type of instabilities which may be encountered.

Stability Evaluation

For a general periodic system, $\dot{x} = Ax$, the stability is obtained by integrating $\dot{\Phi} = A\Phi$, $\Phi(0) = I$ over one period, $t = 0$ to T. Then the eigenvalues of $\alpha = \Phi(0,T)$ determine the stability of the system. The roots on the s-plane are $\Lambda = \frac{1}{T}\ln\Theta$, where Θ is the diagonal eigenvalue matrix of α . Usually it is necessary to numerically integrate the equation for Φ , but an analytical solution is possible here since the periodicity in the coefficients is only due to the impulse sequences.

Hence

$$\dot{\Phi} = \begin{bmatrix} F & GC \\ (\Sigma\delta)A^{-1}BH & (\Sigma\delta)(I - A^{-1}) \end{bmatrix} \Phi$$

has the solution

$$\Phi = \begin{bmatrix} e^{Ft} + (e^{Ft}-I)F^{-1}GCBH & (e^{Ft}-I)F^{-1}GCA \\ BH & A \end{bmatrix}$$

over the first period, $t = 0$ to T. Then the stability of the combined system is defined by the eigenvalues of the matrix

$$\alpha = \Phi(T) = \begin{bmatrix} e^{FT} + (e^{FT} - I)F^{-1}GCBH & (e^{FT} - I)F^{-1}GCA \\ BH & A \end{bmatrix}$$

The roots on the s-plane are $\Lambda = \frac{1}{T} \ln \alpha$, and the eigenvectors may be evaluated from $(PS) = \Phi\, Se^{-\Lambda t}$.

It may be verified that this result has the correct limits for the cases of an entirely continuous system or an entirely discrete system; and for the limit of infinite sampling rate, $T \to 0$. This solution may also be compared with the results obtained by approximating the whole system as all continuous or all discrete. For the discrete case, the continuous plant is approximated by

$$x(k+1) = e^{FT}x(k) + (e^{FT} - I)F^{-1}Gv(k)$$

So the stability is given by the eigenvalues of the matrix

$$\alpha = \begin{bmatrix} e^{FT} & (e^{FT} - I)F^{-1}GC \\ BH & A \end{bmatrix}$$

This approximate model is correct at the sampling times, but contains no information about the behavior of the plant between sampling times, hence the differences in the two eigenvalue problems.

Type of Instabilities

Besides the usual real and complex-conjugate-pair roots on the s-plane, the periodic system may also have pairs of roots which have their frequency fixed at a multiple of π/T. The root locus will show one of these roots becoming less stable and the other more stable (so they are not conjugates). The property of the solution that allows such behavior is that the eigenvectors are periodic. Further discussion of the properties of periodic systems may be found in the literature (such as [1]). Such an instability of a periodic system is analogous to an instability on the real z-axis of a discrete system. The motion is at a frequency $n(2\pi/T)$ if the root is on the positive real axis, and at $(n+\frac{1}{2})(2\pi/T)$ for the root on the negative real axis. This analogous behavior is the basis for the representation of the discrete system by an equivalent periodic, continuous system. As expected, an instability of the combined discrete and continuous system is possible with the frequency locked at a multiple of the Nyquist frequency π/T, and periodic system theory gives the details of such behavior.

Conclusion

A periodic, continuous model has been found which is equivalent to a discrete system, in the sense that it has the same solution at the sampling times, and is constant in between. This allows the formulation of the combined continuous plant and discrete controller as a continuous, periodic system. Then periodic system theory may be applied to analyze the stability and response.

Thus the calculation of the exact stability of combined continuous and discrete systems has been reduced to a matrix eigenvalue problem. Periodic system theory also gives some information on the type of instabilities which may be encountered. The calculation of the exact forced response using periodic system theory has not been considered here, but it may be obtained if it proves necessary.

References

[1] DeRusso, Paul M.; Roy, Rob J.; and Close, Charles M.; State Variables for Engineers, John Wiley and Sons, Inc., New York, 1965